Statistical Intervals Vive La Différence!

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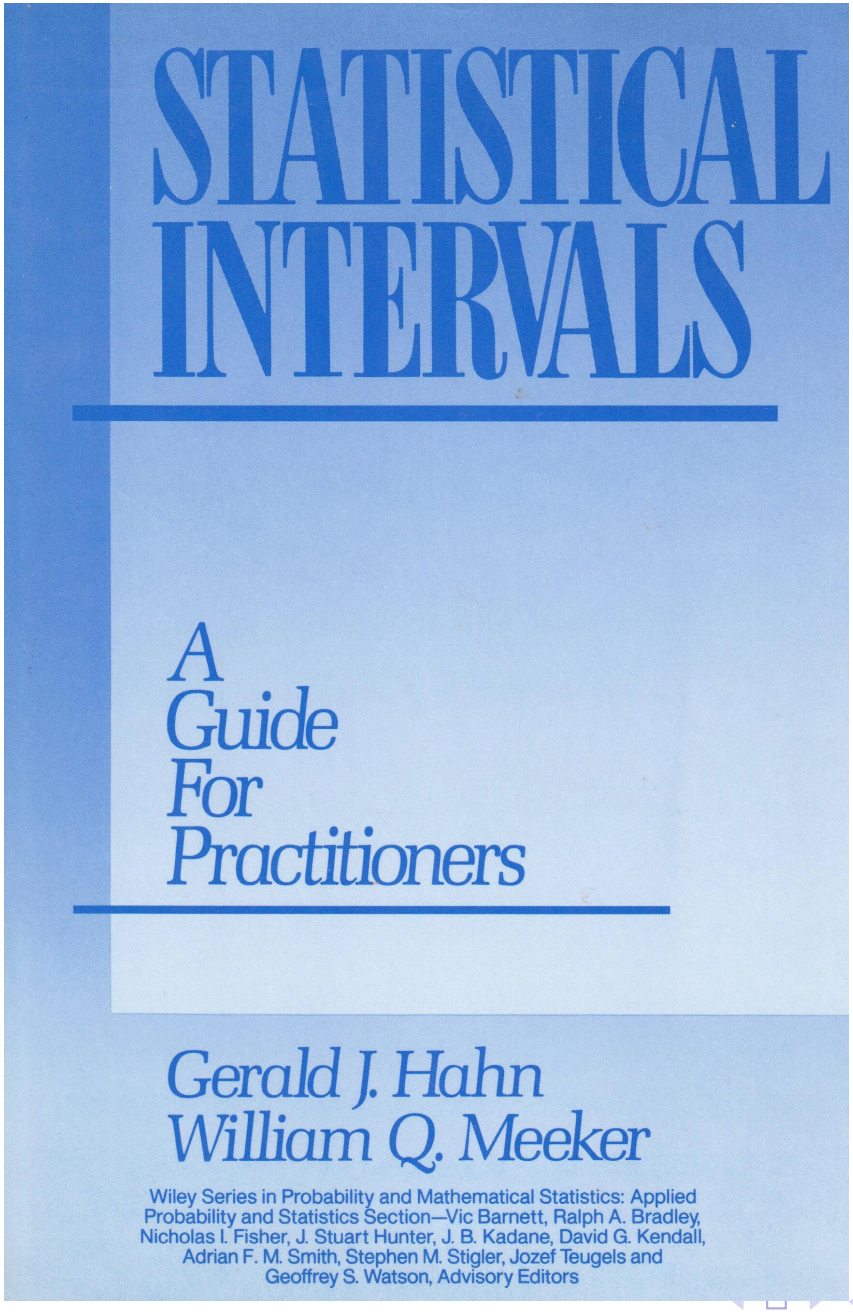
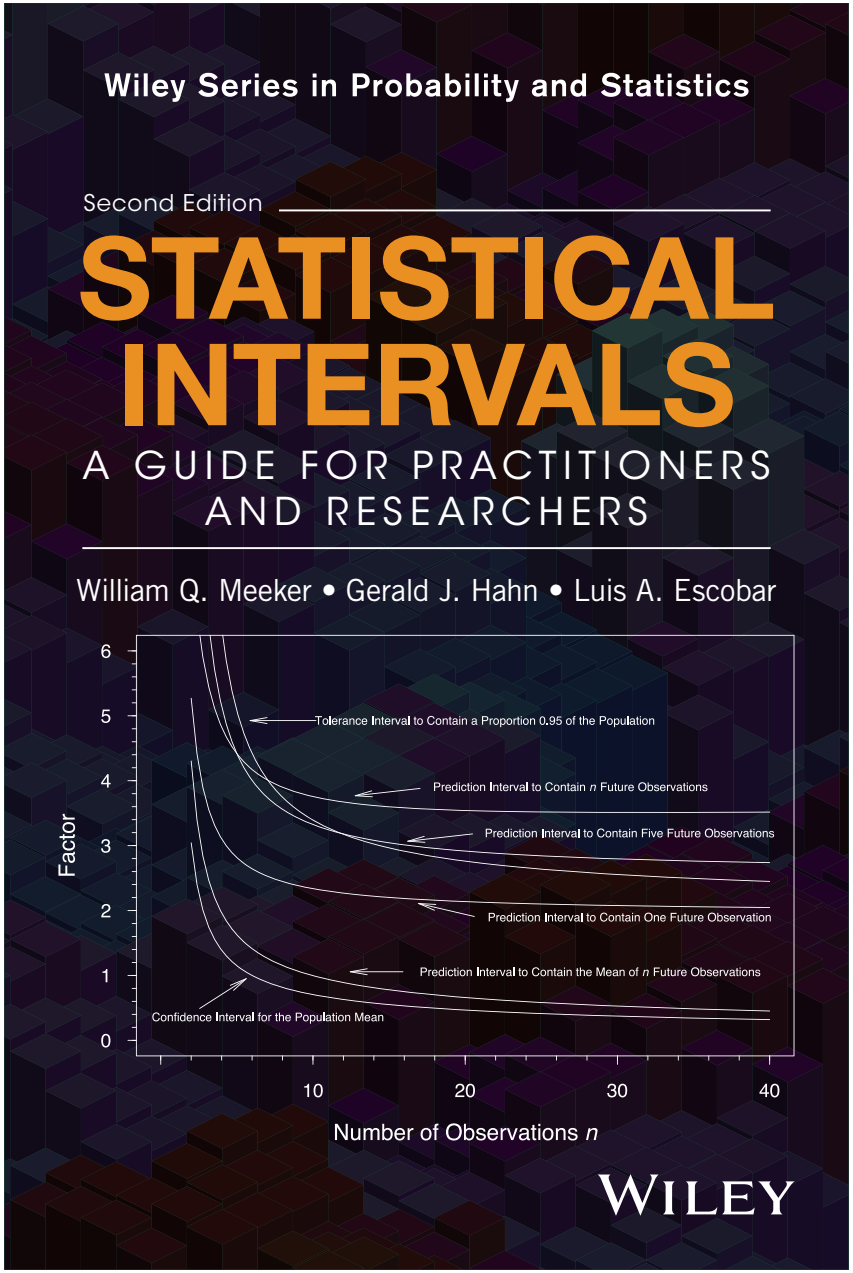
The Different Kinds of Statistical Intervals

- Confidence Intervals
- Tolerance Intervals
- Prediction Intervals

Statistical Intervals Wiley 1991



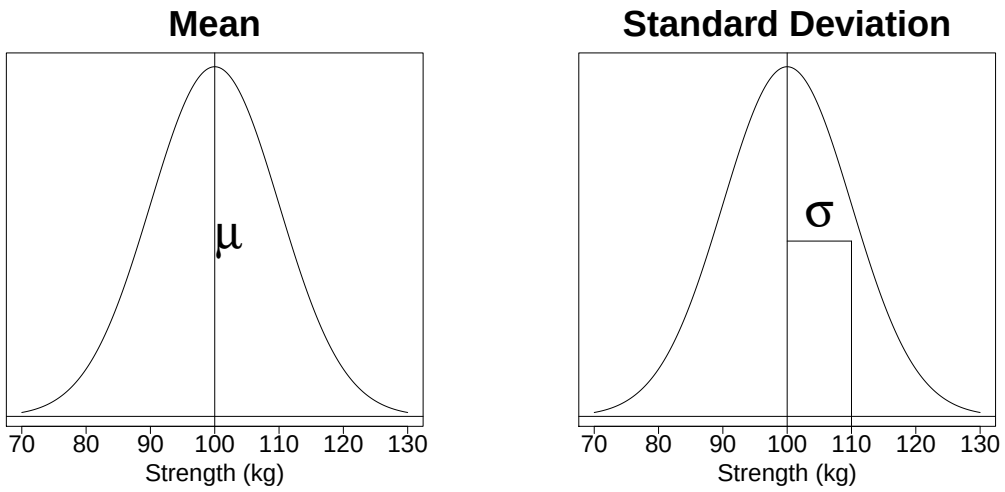
Statistical Intervals Second Edition Wiley 2017



Overview

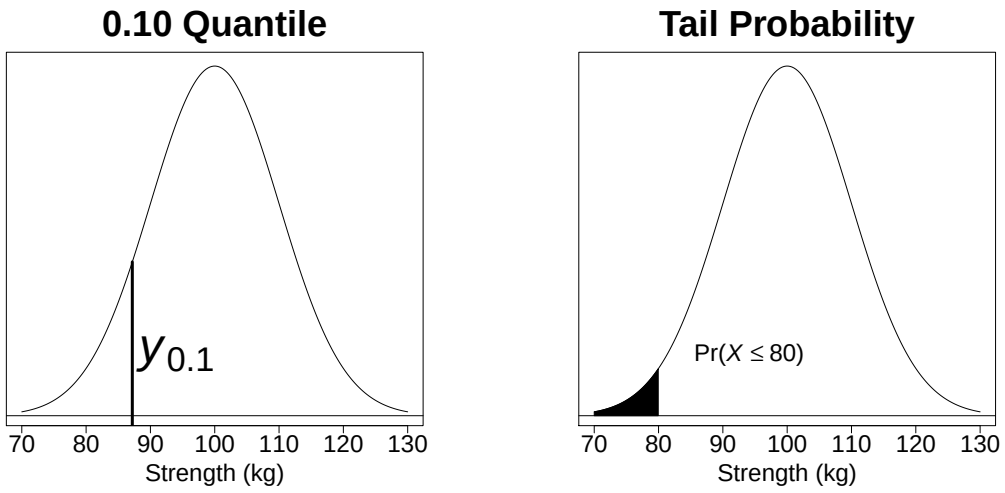
- Confidence Intervals and the “Other Intervals”
- What was in Hahn and Meeker 1991?
- Statistical Intervals: What has changed in the past 25 years?
- Statistical Intervals: Exact, Conservative, and Approximate
- Why did we write *Statistical Intervals*, Second Edition?
- Advances described in *Statistical Intervals*, Second Edition
  - General methods for computing statistical intervals
  - Generalized pivotal quantities
  - Advanced case studies further illustrating general methods
  - Technical appendices
- Predictions about future developments for statistical intervals (Third edition!?)
- Concluding remarks

Confidence Intervals



- Confidence intervals on normal distribution mean and standard deviation routinely covered in elementary text books

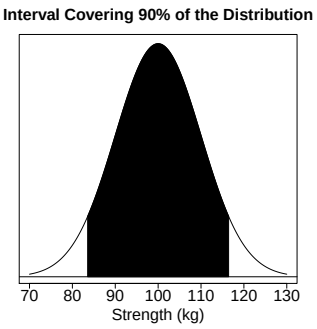
The Other Confidence Intervals



Confidence intervals on **quantiles** and **tail probabilities**

- Frequently needed in applications
- Usually not covered in text books! **Why?**

Tolerance Intervals to Characterize a Distribution



- Tolerance intervals discussed only in a few texts—but frequently needed in practice.
- When distribution parameters are **known**, it is easy to compute a **Probability Interval** that will contain a specified proportion (e.g., $\beta = 0.90$) of the distribution.
- When the parameters are **unknown**, one can compute a statistical **Tolerance Interval** to contain at least a proportion β with $100(1-\alpha)\%$ confidence (e.g., to contain a proportion 0.90 with 95% confidence).
- A **one-sided tolerance bounds** is equivalent to a **one-sided confidence bound** on a distribution **quantile**.

Prediction Intervals

- Prediction Intervals are used to quantify the uncertainty when predicting a **future value of a random variable.**
- Prediction Intervals are well known (e.g., covered in text books) in regression and in time series.
- Prediction Intervals are not so well known in other area of application and infrequently considered in texts, despite their wide applicability. (Consequence: CIs are often incorrectly calculated when PIs are required)
- **Simultaneous** Prediction Intervals to contain k-out-of-m future random variables are sometimes needed.
- When distribution parameters are known, it is easy to compute a **Probability Interval** that will contain a future observation with a specified probability. Otherwise a **statistical** approach is required.

Relationship Between Tolerance and Prediction Intervals

- Tolerance and Prediction Intervals (and when to use them) are **sometimes confused**.
- A k-out-of-m **Simultaneous Prediction Interval** with large m can be approximated by a **tolerance interval** to contain a proportion $\beta = k/m$ of the distribution.
- Tolerance Intervals are appropriate when one wants to describe a distribution.
- Prediction Intervals are appropriate when one wants an interval to contain one or a small number of future random outcomes (e.g., a consumer who buys a single refrigerator).

Outline
Statistical Intervals First Edition

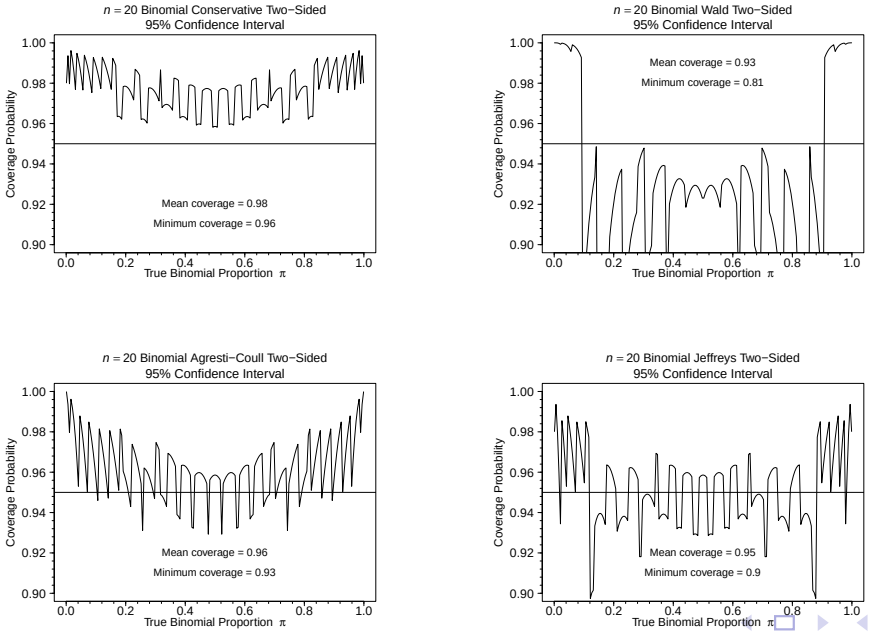
- Background and Assumptions (Chapters 1 and 2)
- Confidence, tolerance, and prediction intervals and examples for
  - Normal distribution (Chapters 3 and 4)
  - Binomial distribution (Chapter 6)
  - Poisson distribution (Chapter 7)
  - Distribution-free intervals (Chapter 5)
- Sample size determination for statistical intervals (Chapters 8-10)
- Basic case studies

Statistical Intervals: What has Changed in the Past 25 Years?

- Recognition that the commonly used text book formulas for Binomial and Poisson distributions confidence intervals are seriously flawed and the development of improved methods
- Wide recognition that likelihood-based intervals are better than Wald (a.k.a., normal-approximation) intervals (and implementation in more commercial statistical software)
- Wider use of bootstrap and simulation-based interval procedures (e.g., general procedures now implemented in JMP Pro)
- A revolution in the use of Bayesian method (and associated intervals) in many practical applications
- Vastly more computational power and ease of doing computations (e.g., using R)

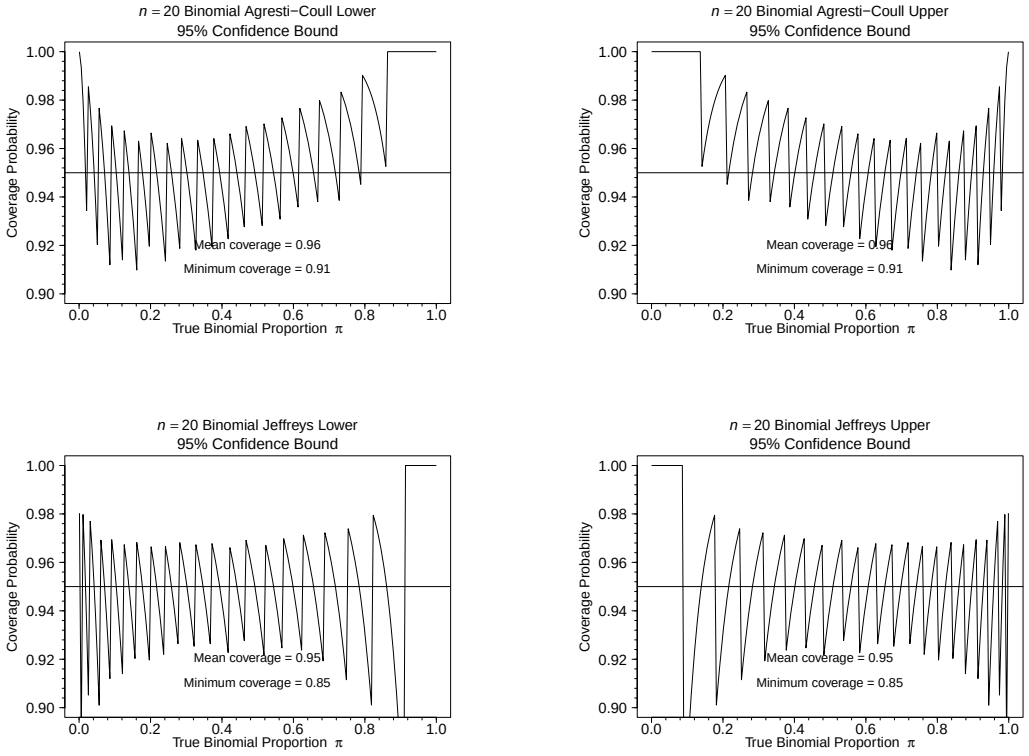
Statistical Intervals: Exact, Conservative, Approximate

- Outside of simple situations, **exact** statistical intervals are usually not available.
- A statistical interval procedure should be evaluated relative to its **coverage probability** (how close it is to the **nominal confidence level using separate evaluations for each tail**) and (secondarily) the expected width or other measure of precision.



Upper/Lower Balance in Error Probabilities is Desirable

Coverage probabilities for lower and upper one-sided confidence bounds



Why Did We Write *Statistical Intervals* Second Edition?

- Bring discussion of statistical interval procedures up to date
- Present **general** methods for computing statistical intervals
- Provide **technical justification** for all intervals (mostly in technical appendices)
- More applications

Advances
Statistical Intervals Second Edition

- New methods for obtaining binomial, Poisson and distribution-free intervals (Chapters 5 to 7 completely re-written)
- Five completely new chapters on **general methods**
  - Likelihood and Wald methods
  - Bootstrap and simulation methods (also pivotal quantities and generalized pivotal quantities)
  - Bayesian methods
  - Hierarchical Models via Bayesian methods
  - Advanced Case Studies and examples
- References and further information are presented in a Bibliographic Notes section at the end of each chapter
- Book is software neutral, but uses R as an advanced calculator to compute certain intervals

Pivotal Quantity (PQ): Confidence Interval for the Mean of a Normal Distribution

The pivotal quantity random variable

$$\frac{\sqrt{n}(\bar{X}-\mu)}{S}$$

has a distribution that does not depend on unknown parameters.

$$\Pr\left[-t_{(1-\alpha/2;n-1)} \leq \frac{\sqrt{n}(\bar{X}-\mu)}{S} \leq t_{(1-\alpha/2;n-1)}\right] = 1-\alpha$$

Solving for μ, gives

$$\Pr\left[\bar{X} - t_{(1-\alpha/2;n-1)})\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{(1-\alpha/2;n-1)}\frac{S}{\sqrt{n}}\right] = 1-\alpha$$

Thus

$$[\underset{\sim}{\mu}, \quad \tilde{\mu}] = \bar{x} \mp t_{(1-\alpha/2;n-1)}\frac{s}{\sqrt{n}}$$

is a $100(1-\alpha)\%$ confidence interval for μ.

Generalized Pivotal Quantities (GPQ)

Given the data, a GPQ Z_θ is a random variable with a distribution that does not depend on unknown parameters (but may depend on the data through MLEs $\widehat{\mu}$ and $\widehat{\sigma}$. For example, GPQs for μ and σ are

$$Z_\mu = \widehat{\mu} + \left(\frac{\mu - \widehat{\mu}^*}{\widehat{\sigma}^*}\right)\widehat{\sigma}$$

$$Z_\sigma = \left(\frac{\sigma}{\widehat{\sigma}^*}\right)\widehat{\sigma}$$

Lower tail probability: $p = F(x) = F(x; \mu, \sigma) = \Phi\left(\frac{x - \mu}{\sigma}\right)$

$$\widehat{p} = \Phi\left(\frac{x - \widehat{\mu}}{\widehat{\sigma}}\right)$$

Then a GPQ for p is

$$Z_p = \Phi\left[\left(\frac{\widehat{\sigma}^*}{\sigma}\right)\Phi^{-1}(\widehat{p}) + \frac{\widehat{\mu}^* - \mu}{\sigma}\right]$$

GPQ Confidence Interval for a Lower Tail Probability

A GPQ for p is

$$Z_p = \Phi\left[\left(\frac{\widehat{\sigma}^*}{\sigma}\right)\Phi^{-1}(\widehat{p}) + \frac{\widehat{\mu}^* - \mu}{\sigma}\right]$$

Simulate a large number (e.g., 100,000) of realizations Z_p^* from the distribution of Z_p. Without loss of generality, can set $\mu = 0$ and $\sigma = 1$ and simulate from

$$Z_p^* = \Phi\left[\widehat{\sigma}^*\Phi^{-1}(\widehat{p}) + \widehat{\mu}^*\right]$$

A $100(1-\alpha)\%$ confidence interval for p is obtained from the $\alpha/2$ and $1-\alpha/2$ quantiles of the empirical distribution of Z_p^*.

Further Comments about GPQs

- Started with work by Tsui and Weerahandi in 1989
- GPQ intervals may give exact interval procedures, but in general provide interval procedures that are asymptotically exact
- GPQ procedures are especially simple for functions of the parameters of location-scale/log-location-scale distributions.
- Some difficult confidence interval problems easily handled by GPQs are
  - Mean of lognormal or Weibull distributions
  - Functions of variance components in GR&R studies
  - Probability of being in a specified interval
  - Two-sample comparisons with different spread or shape parameters
- GPQ inference is related to a generalized form of **Fisher's fiducial inference** (e.g., Hannig, Iyer, and Patterson 2006 and Hannig 2009), providing a theoretical basis for the methods.

Advanced Case Studies Further Illustrating General Methods

- Proportion of defective integrated circuits (LFP model using Wald, likelihood, and bootstrap methods)
- Components of variance in a measurement process (Gauge R&R using Bayes and GPQ)
- Tolerance interval to characterize the distribution of process output in the presence of measurement error (Bayes and GPQ)
- Estimating the proportion of nonconforming product—probability of being between specification limits (Bayes and GPQ)
- Estimating the treatment effect in a marketing campaign (Bayes and simulation)
- Estimating probability of detection with limited hit-miss data (likelihood and Bayes)
- Using prior information to estimate the service-life distribution of a rocket motor (Bayes)

Technical Appendices

- **A.** Notation and Acronyms
- **B.** Generic Definition of Statistical Intervals and Formulas for Computing Coverage Probabilities
- **C.** Important Probability Distributions
- **D.** General Results from Statistical Theory and Some Methods Used to Construct Statistical Intervals
- **E.** Pivotal Methods for Constructing Parametric Statistical Intervals
- **F.** Generalized Pivotal Quantities
- **G.** Distribution Free Intervals Based on Order Statistics
- **H.** Basic Results from Bayesian Inference Models
- **I.** Probability of Successful Demonstration
- **J.** Tables

Predictions for Future Developments and Material for *Statistical Intervals* **Third Edition**

- Additional computing power (but will growth slow?)
- Likelihood methods (perhaps including second-order corrections) will be widely deployed in statistical software
- Bayesian methods will be widely deployed in some advanced statistical software
- Better methods will be developed to specify diffuse prior distributions in Bayesian methods to provide good frequentist properties (**objective Bayes**)
- Continued development of theory relating GPQ methods and generalized fiducial methods
- Possible theoretical unification of Bayesian and non-Bayesian methods

Concluding Remarks

- It is important to quantify statistical uncertainty and statistical intervals provide the best way to do that
- It is critically important that analysts use **appropriate** statistical intervals
- In the use of any statistical inference method, it is important to pay careful attention to **important assumptions** (both those that can be checked and those that cannot)
- Statistical intervals provide much better insight than inference methods like hypothesis testing and *p*-values (e.g., size effect and information about practical significance). See recent ASA statement on the use of *p*-values.
- Improvements in computing power and theoretical developments have provided a clear path to the construction of **appropriate** Statistical Intervals for almost

The End

Thank You